

Semidomains and Metabelian Product of Metabelian Lie Algebras

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1 Introduction

This paper is the third in a series of papers, the aim of which is to construct algebraic geometry over metabelian Lie algebras. The foundations of this theory were layed in the papers [4, 5].

Let A be a metabelian Lie algebra over a field k ; $S(x) = 0$ be a system of equations over A with the set of variables $X = \{x_1, \dots, x_n\}$ and $V(S)$ be the algebraic set from A^n defined by system S . Analysis of proofs of theorems from [5], that establish the structure of algebraic sets $V(S)$ and the structure of their coordinate algebras $\Gamma(S)$ shows that all major calculations are done in $A[X] = A * F(X)$. Here $F(X)$ stands for the free metabelian Lie algebra and $*$ denotes the metabelian product. This argument explains our interest to the problem of investigation of structure of metabelian product of Lie algebras. The solution to this problem is given by Theorems 1, 2 and 4.

The notion of a *domain* for groups (the groups that have no zero divisors) was introduced in [7], where the authors point out its importance for estimation of some certain criterions of irreducibility of algebraic sets. Every Lie algebra or group that possesses non-trivial abelian ideals (subgroups) has zero divisors, thus in the categories of Lie algebras and metabelian Lie algebras the notion of a domain fails (in the case of metabelian Lie algebras the

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commutant is an abelian ideal). These circumstances make us to introduce the notions of a *semidomain* and a *strict semidomain* and investigate their properties in order to apply obtained results to algebraic geometry.

2 Preliminaries

Below we provide the reader with a brief overview of some auxiliary facts on Lie algebra (the basics for Lie algebras theory can be found in [2]).

Recall, that a Lie algebra A over a field k is termed metabelian if and only if the following universal axiom holds:

- $(a \circ b) \circ (c \circ d) = 0$.

By $a \circ b$ or ab we shall denote the product of elements from algebra A . *Left normed products* $a_1 a_2 a_3 \cdots a_n$, of $a_1, a_2, a_3, \dots, a_n \in A$ are defined as

$$(\dots((a_1 \circ a_2) \circ a_3) \circ \dots) \circ a_n.$$

We term such products by *left normed words or monomials* of the *degree* or of the *length* n .

It is well-known that every monomial of the length l from letters a_1, \dots, a_n can be written as a linear combination of left normed monomials of length l from the same set of letters.

Let A be an arbitrary Lie algebra over k and let $\langle x \rangle$ denote the principal ideal generated by $x \in A$. In [2] it is shown, that $\langle x \rangle$ is a k -linear span of left normed words from A , that begin with x :

$$x, xa_1, xb_1b_2, \dots, xc_1c_2 \cdots c_n, \dots$$

Ideal generated by $\{a \circ b | a, b \in A\}$ is called the *commutant* and is denoted by A^2 . The *Fitting's radical* of the Lie algebra A is an ideal, generated by the set of all elements from nilpotent ideals of A . We'll denote the Fitting's radical of the algebra A by $\mathbf{Fit}(A)$. The Fitting's radical can be characterized as follows: an element x from A is an element of $\mathbf{Fit}(A)$ if and only if the ideal $\langle x \rangle$ is nilpotent (see [4]).

Let F be a free algebra in the variety \mathcal{A}^2 of all metabelian Lie algebras. And let $\{a_\alpha | \alpha \in \Lambda\}$ be a free base for F , here Λ is a totally ordered set. Then left normed monomials $a_{i_1} a_{i_2} \dots a_{i_m}$ are called *normalised* if and only if the following condition is satisfied:

$$i_1 > i_2 \leq i_3 \leq \dots \leq i_m.$$

In [1] it is proven that the set of all normalised words form a linear basis of F .

Let A be an arbitrary metabelian Lie algebra over k . Then the commutant A^2 admits a structure of an R -module, here $R = k[X]$ is the ring of polynomials over k and X is a linear basis of vector space A/A^2 (see [4] for details).

3 Metabelian Products of Metabelian Lie Algebras

Let A and B be two metabelian Lie k -algebras. Throughout this paper the field k is fixed, thus throughout by \mathcal{A}^2 we'll assume the variety of all metabelian Lie k -algebras. Via $A * B$ we denote the product of A and B in the variety \mathcal{A}^2 . Recall the definition of the algebra $A * B$.

Definition 1 ([2]) *Let*

$$A = \langle z_i, i \in I_A | R_A \rangle_{\mathcal{A}^2}, \quad B = \langle z_i, i \in I_B | R_B \rangle_{\mathcal{A}^2}$$

*be the presentations of respective algebras in the variety \mathcal{A}^2 . Assume that $I_A \cap I_B = \emptyset$. Then the algebra $A * B$ is defined by the presentation*

$$A * B = \langle z_i, i \in I_A \cup I_B | R_A \cup R_B \rangle_{\mathcal{A}^2}.$$

We also give the categorial definition of $A * B$, which is equivalent to Definition 1.

Definition 2 (Categorial Definition) *A metabelian Lie algebra $C \in \mathcal{A}^2$ is termed metabelian Lie product of metabelian Lie algebras A and B if it satisfies two following conditions*

1. *There exist two injections $i_A : A \rightarrow C$ and $i_B : B \rightarrow C$ such that the algebra C is generated by the images $i_A(A)$ and $i_B(B)$, $C = \langle i_A(A), i_B(B) \rangle_{\mathcal{A}^2}$;*
2. *For every metabelian Lie algebra H over the field k and every pair of injections $\psi_A : A \rightarrow H$ and $\psi_B : B \rightarrow H$ there exists a homomorphism $\phi : C \rightarrow H$ such that $\phi i_A = \psi_A$ and $\phi i_B = \psi_B$.*

In this section we shall investigate the structure of $A * B$ and derive that $\text{Fit}(A * B) = (A * B)^2$.

Let \overline{A} and \overline{B} be the vector spaces over k , $\overline{A} = A/A^2$ and $\overline{B} = B/B^2$. Assume that $X = \{x_i | i \in I\}$ and $Y = \{y_j | j \in J\}$ are the linear bases of \overline{A} and \overline{B} correspondingly and that the sets I and J are totally ordered. For the sake of convenience, by the elements of the sets X and Y we also denote their fixed preimages in A and B .

As we have already mentioned above the commutant A^2 admits the structure of a module over $R_X = k[X]$ and the commutant B^2 the structure of a module over $R_Y = k[Y]$. The k -vector space $A * B / (A * B)^2$ is generated by the set $X \cup Y$. This implies that $(A * B)^2$ can be treated as a module over the ring of polynomials $R_{X \cup Y} = k[X \cup Y]$.

We estimate the structure of the algebra $A * B$ in two steps.

Step 1 Define a 3-tuple of $R_{X \cup Y}$ -modules M_0 , M_1 and M_2 . With the help of these modules we construct a new $R_{X \cup Y}$ -module M .

Step 2 By the module M we construct a metabelian Lie algebra C and prove that C is isomorphic to $A * B$. This isomorphism yields to quite satisfactory structural results for $A * B$.

Extend the modules A^2 and B^2 (R_X - and R_Y - modules respectively) to $R_{X \cup Y}$ -modules by setting:

$$M_1 = A^2 \otimes_{R_X} R_{X \cup Y}, \quad M_2 = B^2 \otimes_{R_Y} R_{X \cup Y},$$

where here ' \otimes_{R_X} ' is the sign of tensor product over R_X and ' \otimes_{R_Y} ' stands for tensor product over R_Y . Let M_0 be a module isomorphic to the ideal of $R_{X \cup Y}$ generated by $\{x_i y_j | i \in I, j \in J\}$. In particular, M_0 is torsion free. Let

$$A^2 = \langle Z_1 | R_1 \rangle; \quad B^2 = \langle Z_2 | R_2 \rangle, \quad Z_1 \cap Z_2 = \emptyset$$

be the presentations of R_X -module A^2 and R_Y -module B^2 . To define the module M by generators and relations we denote some of the generators by a pair of letters. Let $XY = \{w_{ij} \equiv x_i y_j | i \in I, j \in J\}$, $Z = Z_1 \cup Z_2$. We set

$$M = \langle XY \cup Z | R_1 \cup R_2 \cup S \rangle, \tag{1}$$

where here

$$S = \left\{ \begin{array}{l} (x_i y_{j_1}) \cdot y_{j_2} = (x_i y_{j_2}) \cdot y_{j_1} + (y_{j_2} y_{j_1}) \cdot x_i; j_1 > j_2; \\ (x_{i_1} y_j) \cdot x_{i_2} = (x_{i_2} y_j) \cdot x_{i_1} + (x_{i_1} x_{i_2}) \cdot y_j; i_1 > i_2. \end{array} \right.$$

The symbol ‘ \cdot ’ denotes the module action and $y_{j_2}y_{j_1}$, $x_{i_1}x_{i_2}$ denote the Lie product of respective elements in B and A .

Lemma 1 *In this notation,*

1. *the submodule $M_3 = \langle Z_1 \cup Z_2 \rangle$ of M is isomorphic to $M_1 \oplus M_2$;*
2. *the factor-module M/M_3 is isomorphic to M_0 .*

Proof. To prove these statements we shall introduce the structure of $R_{X \cup Y}$ -module on the vector space $M' = M_0 \oplus M_1 \oplus M_2$ in such a way, that the vector space $M_1 \oplus M_2$ becomes a submodule of M' and the induced action on $M_1 \oplus M_2$ coincides with the initial one. Therefore, we only are to introduce the action of $R_{X \cup Y}$ on the additive base of M_0 in the correct way. Recall, that M_0 is isomorphic to the ideal of $R_{X \cup Y}$ generated by the set $\{x_i y_j | i \in I, j \in J\}$. Consequently, its additive base is the set of all monomials from elements of $X \cup Y$ which involve letters from both of X and Y . Let W be the set of all such monomials written lexicographically on X and on Y .

Let $w \in W$. Rewrite w in the form

$$w = (x_i y_j) w^*, \quad (2)$$

here i and j are the least indices of letters from X and Y that are involved in w .

Let V be the set of all monomials from $X \cup Y$. We define the module action ‘ \cdot ’ of an element $w \in W$ on $v \in V$ (and denote it $w \cdot v$) using induction on the degree of v . We set

$$w \cdot x_{i_1} = \begin{cases} (x_i y_j) \cdot w^* x_{i_1}, & \text{if } i_1 \geq i; \\ (x_{i_1} y_j) \cdot w^* x_i + (x_i x_{i_1}) \cdot w^* y_j, & \text{if } i_1 < i. \end{cases} \quad (3)$$

Where $m = (x_i x_{i_1}) \cdot w^* y_j$ is an element of M_1 and $x_i x_{i_1}$ is the Lie product in the algebra A .

$$w \cdot y_{j_1} = \begin{cases} (x_i y_j) \cdot w^* y_{j_1}, & \text{if } j_1 \geq j; \\ (x_i y_{j_1}) \cdot w^* y_j + (y_{j_1} y_j) \cdot w^* x_i, & \text{if } j_1 < j. \end{cases} \quad (4)$$

Where $m = (y_{j_1} y_j) \cdot w^* x_i \in M_2$.

We leave the reader to check that Equations (3) and (4) give rise to the required module structure on M' .

By the definition the module M' is generated by the same elements as the module M is (the set coincide as words). We also notice that all the relations from Equation (1) which are true in M hold in M' . Therefore, there exists a homomorphism $\phi : M \rightarrow M'$. We shall show that ϕ is an isomorphism. Consider the natural additive base \mathcal{B} for the module M' . The preimage of \mathcal{B} under ϕ generates the module M . Moreover, all module relations of \mathcal{B} are satisfied in its preimage. Consequently, $\phi(M_3) = M_1 \oplus M_2$ and the homomorphism ϕ turns out to be an isomorphism and Lemma 1 is proven. ■

By the module M we construct a metabelian Lie algebra C . Let V be a vector space over the field k with the base $X \cup Y$ and let $C = V \oplus M$ be the direct sum of vector spaces over k . We next define the multiplication on the vector space C .

For a pair of elements $c_i \in C$, $c_i = (v_i, m_i); i = 1, 2; v_i \in V, m_i \in M$ set

$$(v_1, m_1) \circ (v_2, m_2) = (0, v_1 \cdot v_2 - m_2 \cdot v_1 + m_1 \cdot v_2). \quad (5)$$

Here

$$v_1 v_2 = \begin{cases} x_i x_j & \text{in } A, & \text{if } v_1 = x_i, v_2 = x_j; \\ y_i y_j & \text{in } B, & \text{if } v_1 = y_i, v_2 = y_j; \\ x_i y_j & \text{the element of } M, & \text{if } v_1 = x_i, v_2 = y_j; \\ -x_i y_j & \text{the element of } M, & \text{if } v_1 = y_i, v_2 = x_j; \\ 0, & & \text{if } v_1 = 0 \text{ or } v_2 = 0. \end{cases}$$

If v_1 and v_2 are linear combinations of elements from $X \cup Y$, then the product $v_1 v_2$ is defined using the distributivity law and the equalities above.

Let C be a vector space equipped with the operations introduced above and $U(C) = \{(0, m) | m \in M\}$.

Lemma 2 *The set $U(C)$ is abelian ideal of the algebra C , $C^2 = U(C)$ and C is a metabelian algebra.*

Proof. Follows directly from the definitions of operations on C ■

Lemma 3 *The algebra C is a Lie algebra, i.e. C satisfies the anti-commutativity identity and the Jacoby identity. Therefore C is a metabelian Lie algebra.*

Proof. 1. The anti-commutativity relation immediately follows from (5)

2. Therefore, only the Jacoby identity is at issue. Consider a 3-tuple of elements from C . Let $x = (v_1, m_1)$, $y = (v_2, m_2)$, $z = (v_3, m_3)$. By the definition

$$x \circ y \circ z = (0, (v_1 v_2) v_3 + (m_1 v_2) v_3 - (m_2 v_1) v_3)$$

and

$$x \circ y \circ z + y \circ z \circ x + z \circ x \circ y = (0, (v_1 v_2) v_3 + (v_2 v_3) v_1 + (v_3 v_1) v_2).$$

We now check that

$$(v_1 v_2) v_3 + (v_2 v_3) v_1 + (v_3 v_1) v_2 = 0. \quad (6)$$

To establish equality (6) we need to verify it on the elements from $X \cup Y$ only. Suppose that v_1, v_2 and v_3 are the elements from either X or Y . In which case (6) holds, since the Jacoby identity holds in both A and B . Consider the case when $v_i; i = 1, 2, 3$ lie in both X and Y . Due to the symmetry we may assume that $v_1 = x_{i_1}$, $v_2 = x_{i_2}$, $v_3 = y_j$. Expression (6) takes the form

$$(v_1 v_2) v_3 + (v_2 v_3) v_1 + (v_3 v_1) v_2 = (x_{i_1} x_{i_2}) \cdot y_j + (x_{i_2} y_j) \cdot x_{i_1} - (x_{i_1} y_j) \cdot x_{i_2}.$$

There are three alternatives:

If $x_{i_1} = x_{i_2}$ then Equation (6) obviously holds.

If $x_{i_1} > x_{i_2}$ then $(x_{i_1} y_j) \cdot x_{i_2} = (x_{i_1} x_{i_2}) \cdot y_j + (x_{i_2} y_j) \cdot x_{i_1}$ and (6) follows.

If $x_{i_2} > x_{i_1}$ analogous to the case above.

■

Consider the algebra $A * B$. Clearly, it allows the generating set $X \cup Y \cup Z_1 \cup Z_2$. Categorical Definition 2 of the algebra $A * B$ yields to the existence of the natural homomorphism $\psi : A * B \rightarrow C$.

Theorem 1 *The homomorphism ψ is an isomorphism. Moreover, $\psi((A * B)^2) \cong U(C) \cong M$. The commutant A^2 generates a submodule of $(A * B)^2$ isomorphic to M_1 and B^2 generates a submodule of $(A * B)^2$ isomorphic to M_2 . The factor-module $(A * B)^2 / (M_1 \oplus M_2)$ is isomorphic to M_0 .*

Proof. Consider the natural additive basis of the algebra C and its natural preimage in $A * B$. Direct calculations show, that this preimage in $A * B$ is a set of additive generators for it (see [1]). Therefore, ψ is an isomorphism. Now all other statements of Theorem 1 follow. ■

Theorem 2 *Let A and B be abelian Lie algebras over k thus A is a vector space over X and B is a vector space over Y . Then $R_{X \cup Y}$ -module $(A * B)^2$ is isomorphic to M_0 .*

Proof. The conditions above imply, that $A = \overline{A}$ and $B = \overline{B}$, therefore, Theorem 2 is a direct corollary of Theorem 1 (in this cas $M_1 = M_2 = 0$). ■

Proposition 1 *The Fitting's radical of $A * B$, $A \neq 0, B \neq 0$ coincides with the commutant $(A * B)^2$,*

$$\text{Fit}(A * B) = (A * B)^2$$

Proof. For every metabelian Lie algebra holds its Fittings radical contains its commutant, thus $\text{Fit}(A * B) \supseteq (A * B)^2$. It is sufficient to show the inverse inclusion. Assume the converse $a \in \text{Fit}(A * B)$ and $a \notin (A * B)^2$. Consider $c = x_i y_j$, where $x_i \in X, y_j \in Y$ are chosen to be the least elements from X and Y respectively. Let $a = v_1 + v_2 + m$. Here v_1 is a linear combination of elements from X , v_2 is a linear combination of elements from Y and $m \in (A * B)^2$. The assumption $a \notin (A * B)^2$ implies that $c \circ \overbrace{a \circ \dots \circ a}^{n \text{ times}} \neq 0$. Therefore, since $a, ca \in \langle a \rangle$ we conclude that the ideal $\langle a \rangle$ is not nilpotent and $a \notin \text{Fit}(A * B)$, what derives a contradiction. ■

4 Semidomains

In this section we introduce the notions of a zero divisor, a semidomain, a strict semidomain and establish the criterion which points out when the metabelian product $A * B$ is a semidomain.

Let A be Lie algebra over field k .

Definition 3 *A nonzero element $x \in A$ is termed a zero divisor if and only if there exists $y \in A, y \neq 0$ such that:*

$$\langle x \rangle \circ \langle y \rangle = 0. \tag{7}$$

Definition 3 is symmetric on x and y , thus we say, that x and y is a pair of zero divisors.

In particular, if A is a metabelian Lie algebra, then a pair x, y is a pair of zero divisors if and only if

$$xy = 0; \quad axy = 0 \quad \forall a \in A. \quad (8)$$

Applying the Jacoby identity we see that (8) is symmetric on x and y ,

$$ayx = xya + axy.$$

Example 1 *Let A be a nilpotent Lie algebra of nilpotency class n . Every element $x \in A$ is a zero divisor. Choose y to be an element from the center of A . Then $\text{id}\langle y \rangle$ is a one-dimensional k -vector space with basis $\{y\}$. The pair (x, y) is a pair of zero divisors.*

Example 2 *Let A be an arbitrary Lie algebra over k . Then every element of the Fitting's radical $\text{Fit}(A)$ of the algebra A is a zero divisor. Consider $x \in \text{Fit}(A)$. Then the ideal $\text{id}\langle x \rangle$ is a nilpotent ideal of A . Applying the argument similar to the one of Example 1 the statement follows.*

Example 3 *Let A be a metabelian Lie k -algebra. Suppose that $x, y \in A; x, y \neq 0$ and $xy = 0, x \in A^2$. Then x, y is a pair of zero divisors. Indeed, for every element a of A the product $ay \in A^2$, therefore $ayx = 0$ and Equation (8) is satisfied.*

Denote $D(A)$ the set of all zero-divisors of algebra A with 0.

Let $D(A)$ be the set of all zero-divisors of the algebra A together with 0.

Definition 4 *A Lie algebra A is termed a semidomain if and only if*

$$D(A) = \text{Fit}(A).$$

On behalf of Example 2, $D(A) \supseteq \text{Fit}(A)$ holds for every Lie algebra.

In paper [4] we have introduced the notion of a *matrix metabelian Lie algebra*, which is constructed using a free module over the rings of polynomials. One can check, that every matrix metabelian Lie algebra is a semidomain.

Example 4 (Non-semidomain) Consider a metabelian Lie algebra A given by the following generators and relations

$$A = \langle a_1, a_2, a_3 | a_1 a_2 a_3 = 0 \rangle_{A^2}.$$

Set $x = a_1 a_2$, $y = a_3$. Notice that $x \in A^2$, $y \notin A^2$ and $xy = 0$, which implies that x and y is a pair of zero divisors. Consider non-zero product $a_3 a_1 a_3 a_3 \cdots a_3 \neq 0$. Consequently, the ideal $\langle a_3 \rangle$ is not nilpotent, $y \notin \mathbf{Fit}(A)$ and thus A is not a semidomain.

Definition 5 A Lie algebra A is termed a strict semidomain if and only if:

$$D(A) = A^2.$$

In the event that A is a metabelian Lie algebra we refine this definition. For $\mathbf{Fit}(A) \supseteq A^2$, then $D(A) \supseteq A^2$, which implies that A is a strict semidomain if and only if

- A is a semidomain and
- $\mathbf{Fit}(A) = A^2$.

On behalf of Example 3, if a metabelian Lie algebra is a strict semidomain, then for all $x, y \in A$; $x, y \neq 0$

$$x \in A^2, y \notin A^2 \rightarrow xy \neq 0. \tag{9}$$

In other words, A^2 regarded as module over the ring of polynomials has no torsion under the action of linear polynomials. In that case we say that the module is *linear-torsion free*.

Theorem 3 (Criterion of 'being strict semidomain') Non-zero metabelian Lie algebra A is a strict semidomain if and only if:

- A is not abelian and
- A^2 is linear-torsion free.

Proof. Let A be a strict semidomain. Therefore, for all $x, y \in A$ equation (9) holds. If A is an abelian Lie algebra, then $A^2 = 0$ and $D(A) = A$, thus, the case when A is an abelian Lie algebra derives a contradiction.

Let A be a nonabelian metabelian Lie algebra and (9) holds. We next derive, that A is a strict semidomain. Consider a pair of zero divisors x, y .

Suppose that $x \in A^2$. Then (9) yields to $y \in A^2$. Now suppose that $x \notin A^2$. For A is not abelian and (9) holds there is to exist $z \in A, z \neq 0$ such that $zx \neq 0$. On the other hand, since x, y is a pair of zero divisors we have that $zxy = 0$. Consequently, the pair zx, y is a pair of zero divisors. Applying the argument similar to the one above we conclude that $y \in A^2$. Therefore $x \in A^2$ — a contradiction. ■

Corollary 1 *The commutant A^2 of nonzero metabelian Lie algebra is linear-torsion free if and only if A is either abelian or a strict semidomain.*

Let $A * B$ be the metabelian product of metabelian Lie algebras A and B . We have investigated the structure of this algebra in Section 3, where we have shown that $\text{Fit}(A * B) = (A * B)^2$. This indicates that the properties of ‘being a semidomain’ and ‘being a strict semidomain’ are either true or false for $A * B$ simultaneously.

On behalf of Theorem 1 the structure of $(A * B)^2$ is defined by three $R_{X \cup Y}$ -modules M_0, M_1 and M_2 . The module M_0 is torsion free, while in general M_1 and M_2 are not (for instance, they contain A^2 and B^2 as submodules). Although the following lemma holds:

Lemma 4 *Assume that the commutant A^2 of the algebra A is linear-torsion free. Then the module M_1 is linear-torsion free.*

Proof. Consider $u \in M_1$ and a linear polynomial $f \in R_{X \cup Y}$:

$$f = \sum \alpha_i x_i + \sum \beta_j y_j.$$

Take the summand with the greatest singleton in u , such that it involves the letters of Y only:

$$u = u_1 + a \otimes y_{j_1}^{g_1} \cdots y_{j_t}^{g_t}, \quad 0 \neq a \in A^2$$

Suppose that $\sum \beta_j y_j \neq 0$ and y_{j_n} is the letter with the greatest index and non-zero coefficient among the letters that occur in the linear combination

$\sum \beta_j y_j$. Then in the product $u \cdot f$ the summand $\beta_j a \otimes y_{j_1}^{g_1} \cdots y_{j_t}^{g_t} \cdot y_{j_n}$ is not equal to zero and does not cancel.

Now suppose that $\sum \beta_j y_j = 0$. This implies that $\sum \alpha_i x_i \neq 0$. According to the restrictions of Lemma 4 A^2 is linear torsion free and we obtain:

$$a \cdot \sum \alpha_i x_i \neq 0.$$

Therefore, in the product $u \cdot f$ the summand

$$(a \cdot \sum \alpha_i x_i) \otimes y_{j_1}^{g_1} \cdots y_{j_t}^{g_t}$$

does not cancel.

Thus, $u \cdot f \neq 0$ — q. e. d. ■

Finally we formulate a criterion for $A * B$ to be a semidomain.

Theorem 4 *Let A and B be two non-zero metabelian Lie algebras over k . Then $A * B$ is a semidomain if and only if each of A and B is either abelian or a strict semidomain.*

Proof. As mentioned above, the properties of 'being a semidomain' and 'being a strict semidomain' are equivalent for $A * B$. Since A and B are non-zero, the algebra $A * B$ is non-abelian. Theorem 3 states that $A * B$ is a semidomain if and only if its commutant $(A * B)^2$ is linear-torsion free.

Assume that A is neither abelian nor strict semidomain. Then the commutant A^2 has linear torsion, i.e. there are non-zero elements x and y of A such that $x \in A^2$, $y \notin A^2$ and $xy = 0$. Therefore, $x \in (A * B)^2$, $y \notin (A * B)^2$ what essentially implies that $(A * B)^2$ has linear torsion. Due to Theorem 3 $A * B$ is not a semidomain.

Now assume, that each of A and B is either abelian or a strict semidomain. Thus, the commutants A^2 and B^2 are linear torsion free. We are to show that $(A * B)^2$ is linear torsion free.

Consider $c \in (A * B)^2$, $c \neq 0$ and a linear polynomial $f \in R_{X \cup Y}$:

$$f = \sum \alpha_i x_i + \sum \beta_j y_j.$$

It is sufficient to show, that $c \cdot f \neq 0$.

Since M_0 is a torsion free module $c \notin M_1 \oplus M_2$ we obtain that $c \cdot f \neq 0$. Consequently, let $c \in M_1 \oplus M_2$ and write it as follows:

$$c = u + v, \quad u \in M_1, \quad v \in M_2.$$

Lemma 4 states, that either $u \cdot f \neq 0$ or $v \cdot f \neq 0$. For $M_1 \cap M_2 = \emptyset$ we have $u \cdot f + v \cdot f \neq 0$, i.e. $c \cdot f \neq 0$. ■

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